

Metaplectic Theta Functions and Global Integrals

Solomon Friedberg and David Ginzburg *

Department of Mathematics, Boston College, Chestnut Hill MA 02467-3806, USA

School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 6997801, Israel

To Steve Rallis, in memoriam

Abstract

We convolve a theta function on an n -fold cover of GL_3 with an automorphic form on an n' -fold cover of GL_2 for suitable n, n' . To do so, we induce the theta function to the n -fold cover of GL_4 and use a Shalika integral. We show that in particular when $n = n' = 3$ this construction gives a new Eulerian integral for an automorphic form on the 3-fold cover of GL_2 (the first such integral was given by Bump and Hoffstein), and when $n = 4, n' = 2$, it gives a Dirichlet series with analytic continuation and functional equation that involves both the Fourier coefficients of an automorphic form of half-integral weight and quartic Gauss sums. The analysis of these cases is based on the uniqueness of the Whittaker model for the local exceptional representation. The constructions studied here may be put in the context of a larger family of global integrals which are constructed using automorphic representations on covering groups. We sketch this wider context and some related conjectures.

MSC2010 Subject Classification: Primary 11F70; Secondary 11F27, 11F37, 11F55

Keywords: Metaplectic group, theta representation, Shalika subgroup, Eisenstein series

1 Introduction

In this paper we initiate the study of a certain family of global integrals which are constructed using automorphic representations on covering groups. Beside their

*This work was supported by the US-Israel Binational Science Foundation, grant number 201219, and by the National Security Agency, grant number H98230-13-1-0246 (Friedberg).

intrinsic interest, such constructions have the potential to relate the values of Whittaker coefficients of the theta representations defined on covering groups of general linear groups of different ranks, a sort of metaplectic version of descent. To begin, we explain the general context for these constructions.

Let F be a global field, \mathbb{A} be the adeles of F , and $GL_m^{(n)}(\mathbb{A})$ be an n -fold cover of $GL_m(\mathbb{A})$, defined when F has a full set of n -th roots of unity. (For convenience we shall assume that F has a full set of $2n$ -th roots of unity below.) There is more than one such cover, but the 2-cocycles describing different covers are related by twists, and agree on $SL_m(\mathbb{A})$. Let $\Theta_m^{(n)}$ denote the theta representation on $GL_m^{(n)}(\mathbb{A})$, constructed by Kazhdan and Patterson [10] via residues of Eisenstein series. For $r \geq 2$, let $E_{\Theta_{2r-1}^{(n)}}(g, s, f_s)$ denote the Eisenstein series defined on $GL_{2r}^{(n)}(\mathbb{A})$ obtained by parabolically inducing $\Theta_{2r-1}^{(n)}$. Since the representation $\Theta_{2r-1}^{(n)}$ is associated with the unipotent orbit indexed by the partition $((r+1)(r-2))$, this Eisenstein series is associated with the unipotent orbit $((r+2)(r-2))$.

Let U_{2r} denote the unipotent radical of the parabolic subgroup of GL_{2r} whose Levi part is GL_2^r , and $\psi_{U_{2r}}$ denote a character of $U_{2r}(F) \backslash U_{2r}(\mathbb{A})$ whose stabilizer inside GL_2^r is the group GL_2 embedded diagonally. The group $U_{2r}(\mathbb{A})$ embeds canonically in $GL_{2r}^{(n)}(\mathbb{A})$ by the trivial section $\mathbf{s}(h) = (h, 1)$. Then the Fourier coefficient

$$\int_{U_{2r}(F) \backslash U_{2r}(\mathbb{A})} E_{\Theta_{2r-1}^{(n)}}(\mathbf{s}(u)g, s, f_s) \psi_{U_{2r}}(u) du$$

is attached to the unipotent orbit (r^2) . Let $Z^n(\mathbb{A})$ denote the group of scalar matrices λI_2 in $GL_2(\mathbb{A})$ such that $\lambda \in \mathbb{A}^\times$ is an n -th power. Then $\mathbf{s}(Z^n(\mathbb{A}))$ is a central subgroup of $GL_2^{(n)}(\mathbb{A})$. Let π denote a cuspidal representation of $GL_2^{(k)}(\mathbb{A})$ for certain k with trivial central character. Then we form the global integral

$$I(\varphi_\pi, s, f_s) = \int_{Z^n(\mathbb{A}) GL_2(F) \backslash GL_2(\mathbb{A})} \int_{U_{2r}(F) \backslash U_{2r}(\mathbb{A})} \varphi_\pi(\mathbf{s}(g)) E_{\Theta_{2r-1}^{(n)}}(\mathbf{s}(ug), s, f_s) \psi_{U_{2r}}(u) du \quad (1)$$

where φ_π is a vector in the space of π . Here k and the choice of cocycles up to twisting are chosen so that the cover splits in the integrand on the embedded GL_2 . (Even though we mod out by $Z^n(\mathbb{A})$ instead of $Z^1(\mathbb{A})$, using the strong approximation theorem it is not difficult to see that the integral converges.)

Experience with integrals involving Eisenstein series and automorphic forms suggests that they are most likely to be Eulerian when the dimension equation is satisfied

(see for example Ginzburg [7].) This is the case here. In other words, for this global integral, we have the identity

$$\dim GL_2 - \dim Z^n + \dim U_{2r} = \dim \pi + \dim E_{\Theta_{2r-1}^{(n)}},$$

where the dimensions on the right-hand side denote the Gelfand-Kirillov dimension as in [7]. Indeed, from the remarks on unipotent orbits above this identity is equivalent to

$$3 + \frac{1}{2} \dim (r^2) = 1 + \frac{1}{2} \dim ((r+2)(r-2)),$$

which is easily verified using, for example [7], Section 2.

In this paper we begin the study of the following

Conjecture 1. *The integral $I(\varphi_\pi, s, f_s)$ has a meromorphic continuation to the full complex plane, and:*

- 1) *If $n < r + 1$, the integral is identically zero for all s .*
- 2) *If $n = r + 1$, the integral is Eulerian, and represents the partial degree two L -function $L^S(\tau(\pi), s)$, where $\tau(\pi)$ is the lift of π to $GL_2(\mathbb{A})$.*
- 3) *If $n > r + 1$ the integral is not Eulerian and represents a certain Dirichlet series which, in the domain $\text{Re}(s) > 1/2$, can have at most a simple pole at $s = \frac{n+1}{2n}$.*

The second part of this conjecture may be regarded as an extension of the philosophy of Bump and Hoffstein [4, 5, 6], who proposed that such an L -function could be obtained by “convolving” φ_π with $\Theta_n^{(n)}$, generalizing the construction of Shimura [13] for $n = 2$. Conjecture 1 indicates a way to construct such L -functions by using theta functions on higher rank groups. Here a “convolution” is an adaption of a Rankin-Selberg integral to the covering group. We note that there is often more than one integral representing a given Langlands L -function, and it is not apparent that all extensions of such integrals to the metaplectic group will give the same results. Also, since φ_π does not typically have a *unique* Whittaker model, any such integral is an example of the class of integrals whose study was initiated by Piatetski-Shapiro and Rallis in their seminal paper [11]. See also Bump and Friedberg [3] who connect the approach of [11] to the Bump-Hoffstein Conjecture, and Suzuki [14] for progress on the Bump-Hoffstein conjecture.

The main result of this paper is an analysis of the global integral (1) in the case when $r = 2$ and $n \leq 4$. We obtain a Dirichlet series for covers of all degrees but not enough is known about the Whittaker coefficients of higher theta functions to compute the series for $n > 4$. However, for $n = 3$ we show that the integral is Eulerian and express the local factors in terms of the Hecke eigenvalues for φ_π . A

different Eulerian integral which is a convolution of φ_π with $\Theta_3^{(3)}$ was first given by Bump and Hoffstein [6]. It is interesting that the integral and resulting Dirichlet series arising from the convolution presented here are fundamentally different from the ones given in [6], but ultimately represent the same L -function.

Let us mention two possible applications of this construction. The first is in the case $n = r + 1$. In this case, Conjecture 1 implies that the integral (1) is a holomorphic function in s . The Eisenstein series has a simple pole at $s = \frac{r+2}{2(r+1)}$, whose residue is the representation $\Theta_{2r}^{(r+1)}$. Therefore replacing the Eisenstein series by $\Theta_{2r}^{(r+1)}$ in (1), we expect to get zero for all choices of data. This suggests that the function

$$f(g) = \int_{U_{2r}(F) \backslash U_{2r}(\mathbb{A})} \theta_{2r}^{(r+1)}(ug) \psi_{U_{2r}}(u) du \quad (2)$$

will not have a constituent which is cuspidal. In fact we have

Conjecture 2. *The automorphic representation of $GL_2^{(r+1)}(\mathbb{A})$ generated by all the functions $f(g)$ is $\Theta_2^{(r+1)}$.*

Note that if we compare the Whittaker coefficients of both sides of (2) we obtain a relation between the Whittaker coefficients of f and the Whittaker coefficients of other theta functions. For example, when $r = 2$, we obtain the following local identity

$$q^{n/6} \overline{\tau_{3,2,f}(p^n)} = \tau_{3,3}(1, p^n) + q^{1/6} G_1^{(3)}(p) \tau_{3,3}(1, p^{n-1})$$

Here $\tau_{3,2,f} = W_f \delta_B^{-1/2}$ denotes the normalized Whittaker coefficient of f at the place p , and the other notations are as in Hoffstein [9]. It follows from the well known formulas for the function $W_{\theta_3^{(3)}}$ (see for example Bump and Hoffstein [4], [5]) that $W_f = W_{\theta_2^{(3)}}$. More broadly, Conjecture 2 implies that the mapping given by (2) is a descent map in the sense of Ginzburg, Rallis and Soudry [8]. Such descent constructions have not been given for general metaplectic covers.

A second situation of interest is $n = r + 2$. In this case the integral should reduce to one involving a theta representation that has a unique Whittaker model, and whose Whittaker coefficients may be computed in terms of n -th order Gauss sums. These integrals then give Dirichlet series with continuation whose coefficients involve both the Fourier coefficients of φ_π and arithmetic data. We illustrate that here when $r = 2$, $n = 4$, and where π is an automorphic representation on the double cover, that is, one corresponding to an automorphic form of half-integral weight. The resulting series involves the Fourier coefficients of φ_π , whose squares are related

to the central values of twisted L -functions, and quartic Gauss sums, and possesses analytic continuation.

The rest of this paper is organized as follows. In Section 2 we set out the notation. Then in Section 3 we introduce the basic integral that we study here: the integral (1) when $r = 2$. This is an integral over the Shalika subgroup of GL_4 . In Section 4, the integral is unfolded and expressed in terms of the Whittaker coefficients of φ_π and $\Theta_3^{(n)}$. This series is then analyzed in Section 5 when $n = 3, 4$. The existence of an Euler product when $n = 3$ is established in Theorem 4.

2 Notation

Fix $n \geq 2$. We work over a number field F containing a full set of $2n$ -th roots of unity. Let S be a finite set of places containing the archimedean ones, the ramified ones, and enough others such that the ring of S -integers \mathcal{O}_S has class number 1. Let $F_S = \prod_{v \in S} F_v$ and embed F in F_S diagonally. Then for $r \geq 1$ $GL_r(\mathcal{O}_S)$ is a discrete subgroup of $GL_r(F_S)$.

The metaplectic covers of $GL_r(F_S)$ were constructed by Matsumoto following the work of Kubota when $r = 2$. Convenient references are Bump-Hoffstein [6] and Kazhdan-Patterson [10]. Recall that the basic cover is constructed from embedding $GL_r(F_S)$ into $SL_{r+1}(F_S)$ via $g \mapsto \begin{pmatrix} g & \\ & \det(g)^{-1} \end{pmatrix}$ and restricting the n -fold cover of $SL_{r+1}(F_S)$. The cocycle giving this cover may then be twisted to obtain covers $\widetilde{GL}_r^{(c)}(F_S)$ where $c \in \mathbb{Z}/n\mathbb{Z}$ (see [10], pg. 41); these groups are distinct but all contain the n -fold cover of $SL_r(F_S)$ constructed by Matsumoto. For convenience we write \tilde{G}_j for the group $\widetilde{GL}_j^{(c)}(F_S)$, or $\tilde{G}_j^{(c)}$ if it is important to identify which of the covers we are using. The group $SL_r(\mathcal{O}_S)$ embeds in \tilde{G}_r by the map $\iota(\gamma) = (\gamma, \kappa(\gamma))$, where κ is the Kubota homomorphism (see for example Brubaker, Bump and Friedberg [1], Section 4).

The reference [10] is adelic while [1] uses the above notation. The two approaches are easily connected. Indeed, for each place v of F let \mathcal{O}_v denote the ring of integers in F_v . Then for primes outside S , the hyperspecial maximal compact subgroup $GL_r(\mathcal{O}_v)$ embeds canonically in the metaplectic group (see for example [10], Prop. 0.1.2). Thus the strong approximation theorem shows that working over F_S is equivalent to working adelically and unramified outside S .

The *theta representation* of concern here, $\Theta_3^{(n)}$ or simply Θ , is a genuine representation on $\tilde{G}_3^{(c)}$. It may be constructed globally from residues of Eisenstein series on $\tilde{G}_3^{(c)}$, and its local constituents Θ_v at almost all places may be obtained as the image of a certain unramified principal series representation under an intertwining

operator. See Kazhdan-Patterson [10] for details. By the results of [10], the representations Θ_v have a unique Whittaker model for $n = 3$ or $n = 4$ (for $n = 4$ this requires that c be odd; see [10] Corollary I.3.6). However, for $n > 4$ the model is no longer unique. By contrast, when $n = 2$ all Whittaker coefficients are zero.

3 The Integral

Let φ be a genuine cuspidal automorphic form on the n' -fold cover $\widetilde{GL}_2^{(c')}(F_S)$ of $GL_2(F_S)$ where

$$2/n + 1/n' \in \mathbb{Z}, \quad 4c + c' \equiv 0 \pmod{n}. \quad (3)$$

We suppose that the finite set S is sufficiently large that Θ and φ are unramified outside S . We now present an integral of φ against an Eisenstein series constructed from Θ . Though we describe the construction for general covers, our main focus will be the cases of covers of degrees $n = n' = 3$ and $n = 4, n' = 2$, as in those cases the Whittaker coefficients of Θ are uniquely identified. Using this identification, we are then able to completely describe the resulting Dirichlet series, which has continuation and functional equation as it arises from an integral of an Eisenstein series.

Let P be the standard parabolic subgroup of GL_4 of type (3,1); the Levi of P is isomorphic to $GL_3 \times GL_1$. Let $E_\Theta(g, s, f_s)$ denote the Eisenstein series on \tilde{G}_4 induced from Θ using a suitable section f_s . This series is the double residue of the minimal parabolic Eisenstein series on \tilde{G}_4 introduced by Kazhdan and Patterson [10], pg. 109; see also Brubaker, Bump and Friedberg [1] where the minimal parabolic Eisenstein series is constructed using similar notation to that given here (and where its Whittaker expansion is computed). The series $E_\Theta(g, s, f_s)$ may be written as a sum

$$E_\Theta(g, s, f_s) = \sum_{\gamma \in P(\mathcal{O}_S) \backslash SL_4(\mathcal{O}_S)} f_s(\iota(\gamma)g). \quad g \in \tilde{G}_4$$

(for example, by taking residues in [1], Eqn. (23)). Here we are taking the central character of φ to be trivial but one could relax this assumption by incorporating a suitable character into the Eisenstein series.

Let M denote the algebraic group of 2×2 matrices and let R denote the Shalika subgroup of GL_4

$$R = \left\{ \begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \mid m \in M, g \in GL_2 \right\}.$$

Fix an additive character ψ of F_S of conductor \mathcal{O}_S . For $h \in \mathcal{O}_S$, the map $\psi_h : R(F_S) \rightarrow \mathbb{C}^\times$ given by

$$\psi_h \left(\begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) = \psi(h \operatorname{tr}(M))$$

is a character of $R(F_S)$ that is trivial on $R(\mathcal{O}_S)$.

Recall that the standard maximal compact subgroup of $GL_2(F_S)$ is $K = \prod_{v \in S} K_v$ where $K_v = GL_2(\mathcal{O}_v)$ if v is nonarchimedean and $K_v = U_2(\mathbb{C})$ if v is archimedean (hence necessarily complex). For GL_r with $r > 1$, let $\mathbf{s} : GL_r(F_S) \rightarrow \tilde{G}_r$ denote the trivial section $\mathbf{s}(g) = (g, 1)$. We suppose that φ and E_Θ have compatible K -types: the function

$$g \mapsto \varphi(g_1 \mathbf{s}(g)) E_\Theta \left(g_2 \mathbf{s} \begin{pmatrix} g & \\ & g \end{pmatrix} \right)$$

is right K -invariant for any g_1, g_2 in the corresponding metaplectic groups.

Let $h \in \mathcal{O}_S$. Then the following integral is well defined:

$$I(\varphi, s, f_s) = \int_{Z^n(F_S)R(\mathcal{O}_S) \backslash R(F_S)} \varphi(\mathbf{s}(g)) E_\Theta \left(\mathbf{s} \left(\begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \right) \psi(h \operatorname{tr}(M)) dM dg.$$

(We suppress the dependence of this integral on h from the notation.) Indeed, the covers match by (3) so the integrand is $R(\mathcal{O}_S)$ -invariant. This is our main object of study.

4 Unfolding

To carry out the unfolding, we suppose that $\operatorname{Re}(s)$ is sufficiently large. Here and below, unless otherwise specified, all elements of $GL_4(F_S)$ are embedded in \tilde{G}_4 via the trivial section \mathbf{s} , and we suppress \mathbf{s} from the notation. We first substitute the Eisenstein series in the form

$$E_\Theta(g, s, f_s) = \sum_{\gamma \in P(\mathcal{O}_S) \backslash SL_4(\mathcal{O}_S)} \kappa(\gamma) f_s(\gamma w_0 g).$$

Here we are introducing w_0 , the long element, for convenience. Then the cosets are parametrized by their bottom rows (D_1, D_2, D_3, D_4) modulo \mathcal{O}_S^\times . The 4-tuple (D_1, D_2, D_3, D_4) is relatively prime, and conversely each relatively prime 4-tuple may be completed to an element of $SL_4(\mathcal{O}_S)$. The cosets $P(\mathcal{O}_S)\gamma$ such that γw_0 is in the

big Bruhat cell are those with $D_4 \neq 0$. As is usual, the cosets not in the big cell contribute zero to the integral and we omit them from now on.

On the cosets with $D_4 \neq 0$, we have a right action of $R(\mathcal{O}_S)$ that allows us to collapse the sum over bottom rows to a sum over γ with bottom row $(D_1, D_2, 0, D_4)$ with D_1 and D_2 modulo D_4 (and D_4 modulo \mathcal{O}_S^\times) and in exchange to unfold the integrals. When we do so, we obtain

$$I(\varphi, s, f_s) = \sum_{\gamma} \kappa(\gamma) \int_{Z^n(F_S) B_-(\mathcal{O}_S) \backslash GL_2(F_S)} \int_{Y(\mathcal{O}_S) \backslash Y(F_S)} \int_{X(F_S)} \varphi(g) f_s \left(\gamma w_0 \begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \psi(h \operatorname{tr}(M)) dM dg. \quad (4)$$

Here X and Y are each copies of the affine plane and we have written $M = \begin{pmatrix} X \\ Y \end{pmatrix}$, and B_- is the lower Borel subgroup of GL_2 .

The sum over γ in (4) may be parametrized as a sum over nonzero $D_4 \in \mathcal{O}_S$, D_4 modulo \mathcal{O}_S^\times , and over $D_1, D_2 \in \mathcal{O}_S$, each modulo D_4 , and such that $\gcd(D_1, D_2, D_4) = 1$. To go farther, we must find coset representatives and compute the Kubota symbol of such a coset representative. To do so we follow the approach of Brubaker, Bump and Friedberg [1], generalized to all parabolics in Brubaker and Friedberg [2], and find coset representatives which are products of embedded GL_2 's. Indeed, we parametrize the cosets by representatives $\gamma = \gamma_1 \gamma_2$ with

$$\gamma_1 = \begin{pmatrix} 1 & & & \\ & a_1 & b_1 & \\ & & 1 & \\ & c_1 & & d_1 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} a_2 & & b_2 & \\ & 1 & & \\ & & 1 & \\ c_2 & & & d_2 \end{pmatrix}.$$

Note that the bottom row of γ is then $(c_2 d_1, c_1, 0, d_1 d_2)$ and the sum will be over d_1, d_2, c_1, c_2 with d_1, d_2 modulo \mathcal{O}_S^\times and nonzero, c_1 modulo $d_1 d_2$, $\gcd(c_1, d_1) = 1$ and c_2 modulo d_2 , $\gcd(c_2, d_2) = 1$. We have $\kappa(\gamma) = \left(\frac{d_1}{c_1}\right) \left(\frac{d_2}{c_2}\right)$ where $(-)$ is the n -th power residue symbol as in [1].

Let $\alpha_1, \alpha_2, \alpha_3$ be the three simple roots of GL_4 corresponding to the standard ordering (so $\alpha_1(\operatorname{diag}(t_1, t_2, t_3, t_4)) = t_1/t_2$, etc.) and for each suitable index set J let $u_J(t)$ be the upper triangular unipotent matrix with t in position corresponding to the root $\alpha_J := \sum_{j \in J} \alpha_j$, and $u_J^-(t)$ be the lower triangular unipotent corresponding to $-\alpha_J$. Let $h_J(t)$ denote the corresponding diagonal matrix in GL_4 , so $\alpha_J(h_J(t)) = t^2$. Applying the Bruhat decomposition to γ_1 and γ_2 , we see that

$$\gamma w_0 = u_{23}(b_1/d_1) h_{23}(d_1^{-1}) u_{23}^-(c_1/d_1) u_{123}(b_2/d_2) h_{123}(d_2^{-1}) u_{123}^-(c_2/d_2) w_0.$$

Embedding each matrix via the trivial section \mathbf{s} , this equality holds for the metaplectic group, provided one adds the factor $(d_1, c_1)_S (d_2, c_2)_S$ on the right-hand side. Here $(\ , \)_S$ is the product of local Hilbert symbols $\prod_{v \in S} (\ , \)_v$. See [1], Eqn. (24). Moreover, by reciprocity for n -th power residue symbols, we have

$$(d_1, c_1)_S (d_2, c_2)_S \kappa(\gamma) = \left(\frac{c_1}{d_1} \right) \left(\frac{c_2}{d_2} \right).$$

Now $u_{23}(b_1/d_1)$ is in the unipotent radical of P and does not affect f_s . And $u_{123}^-(c_2/d_2)w_0 = w_0 u_{123}(c_2/d_2)$. The factor $u_{123}(c_2/d_2) \in R(F_S)$ may be absorbed into the integral by a variable change that does not introduce a factor. We multiply (or, more conceptually, apply the Steinberg relations) to see that

$$u_{23}^-(c_1/d_1)u_{123}(b_2/d_2) = u_{123}(b_2/d_2)u_1(-b_2c_1/d_2d_1)u_{23}^-(c_1/d_1).$$

Now the matrix $u_{23}^-(c_1/d_1)$ may be moved rightward and into $R(F_S)$ and absorbed by a variable change; this introduces a factor of $\psi(hc_1/d_1d_2)$. Moreover, the matrix $u_{123}(b_2/d_2)$ is in the unipotent radical of P and does not affect f_s . And the matrix $u_1(-b_2c_1/d_2d_1)$ differs from an element of R by an element of the unipotent radical of P . That is, we obtain the function f_s evaluated as follows:

$$f_s \left(\begin{pmatrix} d_1^{-1}d_2^{-2} & & & \\ & d_1^{-2}d_2^{-1} & & \\ & & d_1^{-1}d_2^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -b_2c_1/d_1 & & \\ & 1 & & \\ & & 1 & -b_2c_1/d_1 \\ & & & 1 \end{pmatrix} w_0 \right. \\ \left. \begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} d_1d_2 & & & \\ & d_1d_2 & & \\ & & d_1d_2 & \\ & & & d_1d_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right).$$

(In these manipulations we use the formulas of [1], Section 3, and see that no non-trivial Hilbert symbols arise.)

We now move the second matrix in the argument of f_s to the right and change

variables in M, g . This gives

$$\sum_{\substack{0 \neq d_1, d_2 \in \mathcal{O}_S / \mathcal{O}_S^\times \\ c_1 \bmod d_1 d_2, (c_1, d_1)=1 \\ c_2 \bmod^\times d_2}} \left(\frac{c_1}{d_1} \right) \left(\frac{c_2}{d_2} \right) \psi \left(\frac{h c_1}{d_1 d_2} \right) \int \varphi^{[d_1^{-1} d_2^{-1}]} \left(\begin{pmatrix} 1 & \\ b_2 c_1 / d_2 & 1 \end{pmatrix} g \right) \\ f_s \left(\begin{pmatrix} d_1^{-1} d_2^{-2} & & \\ & d_1^{-2} d_2^{-1} & \\ & & d_1^{-1} d_2^{-1} \\ & & & 1 \end{pmatrix} w_0 \begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \psi(h \operatorname{tr}(M)) dM dg.$$

Here and below $a \bmod^\times c$ means a modulo c with $(a, c) = 1$, b_2 is a multiplicative inverse of c_2 modulo d_2 , and for $f \in F^\times$,

$$\varphi^{[f]}(g) = \varphi \left(\mathbf{s} \left(\begin{pmatrix} f & \\ & 1 \end{pmatrix} g \right) \right).$$

For α_j , $j = 1, 2, 3$, a simple root we let w_j be the corresponding simple reflection and let $w_{j_1 j_2} = w_{j_1} w_{j_2}$, etc. Factor $w_0 = w_{21} w_{32} w_{13}$. Then $w_{21} \in P(\mathcal{O}_S)$ and may be moved leftward and out of the function f_s . (A cocycle computation is necessary but in fact no nontrivial Hilbert symbol arises.) Similarly $w_{13} \in R(\mathcal{O}_S)$ and may be moved rightward. After a variable change sending g to $w_{13} g$ we obtain

$$\sum_{\substack{0 \neq d_1, d_2 \in \mathcal{O}_S / \mathcal{O}_S^\times \\ c_1 \bmod d_1 d_2, (c_1, d_1)=1 \\ c_2 \bmod^\times d_2}} \left(\frac{c_1}{d_1} \right) \left(\frac{c_2}{d_2} \right) \psi \left(\frac{h c_1}{d_1 d_2} \right) \\ \int_{Z^n(F_S) B(\mathcal{O}_S) \backslash GL_2(F_S)} \int_{X(F_S)} \int_{Y(\mathcal{O}_S) \backslash Y(F_S)} \varphi^{[d_1^{-1} d_2^{-1}]} \left(\begin{pmatrix} 1 & b_2 c_1 / d_2 \\ & 1 \end{pmatrix} g \right) \\ f_s \left(\begin{pmatrix} d_1^{-1} d_2^{-1} & & \\ & d_1^{-1} d_2^{-2} & \\ & & d_1^{-2} d_2^{-1} \\ & & & 1 \end{pmatrix} w_{32} \begin{pmatrix} I_2 & M \\ & I_2 \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \psi(h \operatorname{tr}(M)) dM dg. \quad (5)$$

Here we now have $M = \begin{pmatrix} Y \\ X \end{pmatrix}$, and B denotes the upper triangular Borel subgroup of GL_2 . Note that after the variable change we have also moved the long element of GL_2 leftward and out of the argument of φ .

The next step is to expand φ in a Fourier expansion. Recall we have chosen S so that φ is K_v -fixed for $v \notin S$. This expansion is of the form

$$\varphi(g) = \sum_{0 \neq m \in \mathcal{O}_S} \sum_l \frac{b_l(m)}{|m|^{1/2}} W_l \left(\begin{pmatrix} m & \\ & 1 \end{pmatrix} g \right). \quad (6)$$

Here $|m|$ denotes the cardinality of $\mathcal{O}_S/m\mathcal{O}_S$, W_l runs over a basis for the *finite-dimensional* space of Whittaker functions (in particular, each function W_l satisfies $W_l \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) = \psi(x)W_l(g)$) and the Fourier coefficients $b_j(m)$ satisfy $b_j(\epsilon m) = b_j(m)$ for all $\epsilon \in \mathcal{O}_S^\times$. See Bump-Hoffstein [6], Eqns. (3.13), (3.14). For later use, we note that if $f \in F^\times$ then the expansion (6) implies that

$$\varphi^{[f]}(g) = \sum_{0 \neq m \in \mathcal{O}_S} \sum_{j,l} X_{lj}^{[f]} \frac{b_l(m)(m, f)_S}{|m|^{1/2}} W_j \left(\begin{pmatrix} m & \\ & 1 \end{pmatrix} g \right),$$

where the complex numbers $X_{lj}^{[f]}$ are defined by

$$W_l \left(\begin{pmatrix} f & \\ & 1 \end{pmatrix} g \right) = \sum_j X_{lj}^{[f]} W_j(g).$$

We now use the Iwasawa decomposition on $GL_2(F_S)$ to replace g in the integral by $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} t$ where x runs over F_S and t runs over the maximal torus $T(F_S)$ of $GL_2(F_S)$ consisting of diagonal matrices. We substitute the Fourier expansion for φ into (5). We then move the Y -unipotents and the matrix in x leftward. Since

$$w_{32}u_1(x)u_3(x) = u_{123}(x)u_2(x)w_{32}$$

and $u_{123}(x)$ is in the unipotent radical of P , the resulting integrand becomes

$$\begin{aligned} & \sum_m \sum_{j,l} X_{lj}^{[d_1^{-1}d_2^{-1}]} \frac{b_l(m)(m, d_1^{-1}d_2^{-1})_S}{|m|^{1/2}} \psi \left(\frac{mb_2c_1}{d_2} \right) W_j \left(\begin{pmatrix} m & \\ & 1 \end{pmatrix} t \right) \\ & f_s \left(\begin{pmatrix} d_1^{-1}d_2^{-1} & & & \\ & d_1^{-1}d_2^{-2} & & \\ & & d_1^{-2}d_2^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_2 \\ & 1 & x \\ & & 1 \\ & & & 1 \end{pmatrix} w_{32} \begin{pmatrix} 1 & & & \\ & 1 & x_1 & x_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t & \\ & t \end{pmatrix} \right) \\ & \psi(mx + h(y_1 + x_2)). \end{aligned}$$

We now recognize the integral over y_1, y_2 and x as being a Whittaker integral of the inducing data. Denote the Whittaker coefficients of the theta function on the

n -fold cover of GL_3 by $\tau(r_1, r_2)$, normalized as in [6], Section 4 (see that reference for more information about the coefficients). Then we have expressed $I(s, \varphi, f_s)$ as a finite sum

$$I(s, \varphi, f_s) = \sum_j D_j(s, \varphi) I_j(s, \varphi, f_s). \quad (7)$$

Here $D_j(s, \varphi)$ is the Dirichlet series

$$D_j(s, \varphi) = |h|^{-1} \sum_{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_{\substack{0 \neq d_1, d_2 \in \mathcal{O}_S / \mathcal{O}_S^\times \\ d_2 | h, d_1 | m d_2 \\ c_1 \bmod d_1 d_2, (c_1, d_1) = 1 \\ c_2 \bmod^\times d_2}} \sum_l \left(\frac{c_1}{d_1} \right) \left(\frac{c_2}{d_2} \right) \psi \left(\frac{h c_1}{d_1 d_2} \right) \psi \left(\frac{m b_2 c_1}{d_2} \right) \\ X_{lj}^{[d_1^{-1} d_2^{-1}]} b_l(m) (d_1 d_2, m)_S \tau(m d_1^{-1} d_2, h d_2^{-1}) |d_1| |m|^{-2s+1/2} |d_1 d_2|^{-4s}.$$

The divisibility conditions on d_1 and d_2 follow from the invariance of f_s under lower triangular matrices in $P(\mathcal{O}_S)$, and we have used that $(m, d_1^{-1} d_2^{-1})_S = (d_1 d_2, m)_S$. The factor that multiplies the series $D_j(s, \varphi)$ is

$$I_j(s, \varphi, f_s) = \int_{X(F_S)} \int_{Z^n(F_S) \backslash T(F_S)} W_j(t) \\ W_{f_s} \left(w_{32} \begin{pmatrix} 1 & & & \\ & 1 & x_1 & x_2 \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} t & \\ & t \end{pmatrix} \right) \psi(h x_2) \delta_B(t)^{-1} dt dx_1 dx_2,$$

where

$$W_{f_s}(g) = \int_{N_3(\mathcal{O}_S) \backslash N_3(F_S)} f_s \left(\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} g \right) \psi(n_{12} + n_{23}) dn,$$

with N_3 the unipotent radical of standard Borel in GL_3 . This function may be expressed in terms of the Whittaker function at places in S for Θ .

Since the Eisenstein series has analytic continuation, we conclude that

Theorem 3. *The function $\sum_j D_j(s, \varphi) I_j(s, \varphi, f_s)$ has analytic continuation to all complex s .*

Of course, since the Eisenstein series has functional equation, the series function $\sum_j D_j I_j$ does too. But we shall not compute the intertwining operators necessary to give this explicitly.

When $n = 2$, Θ does not have a Whittaker model. Since all non-degenerate Whittaker coefficients $\tau(n_1, n_2)$ are zero, equation (7) shows that the integral is zero, confirming Conjecture 1, Part 1, in this case. (When $n = 1$ there is no theta function, but if one used the constant function in its place once again the integral would be zero by the same argument.) We consider the first nonzero cases $n = 3, 4$ in the next Section.

5 Analysis of $D_j(s, \varphi)$ when $n = 3$ or $n = 4$

As mentioned above, Kazhdan and Patterson [10] showed that the theta representation Θ on an n -fold cover of $GL_3(F_S)$ has a unique Whittaker model only when $n = 3$ or $n = 4$. (If $n = 4$ this is only true for certain covers.) It is a difficult problem to determine the Whittaker coefficients $\tau(r_1, r_2)$ of functions in the space of Θ for $n > 4$; even on GL_2 the analogous coefficients are not determined (there for $n > 3$). In this Section we consider the two cases where one does have a unique Whittaker model and use this information to analyze the series $D(s, \varphi)$.

For convenience we suppose that h , which appears in the character ψ_h , equals 1 (working with a more general h only changes the series at the primes dividing h). Since $d_2|h$, we have $d_2 = 1$ as well. The series $D_j(s, \varphi)$ reduces to

$$D_j(s, \varphi) = \sum_{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_{\substack{0 \neq d_1 \in \mathcal{O}_S / \mathcal{O}_S^\times \\ d_1 | m}} \sum_l g(d_1) (d_1, m)_S X_{lj}^{[d_1^{-1}]} b_l(m) \tau(md_1^{-1}, 1) |m|^{-2s+1/2} |d_1|^{1-4s},$$

where $g(d_1)$ is the Gauss sum (for \mathcal{O}_S)

$$g_t(d_1) = \sum_{c_1 \bmod^\times d_1} \left(\frac{c_1}{d_1} \right)^t \psi \left(\frac{c_1}{d_1} \right)$$

with $t = 1$. After an interchange of summation (and using properties of the Hilbert symbol and also replacing d_1 by simply d), we obtain

$$D_j(s, \varphi) = \sum_{0 \neq d \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_l X_{lj}^{[d^{-1}]} g(d) (m, d)_S b_l(md) \tau(m, 1) |m|^{-2s+1/2} |d|^{3/2-6s}. \quad (8)$$

We recall that the Gauss sum $g(d)$ satisfies the equation

$$g(d_1 d_2) = \left(\frac{d_1}{d_2} \right) \left(\frac{d_2}{d_1} \right) g(d_1) g(d_2)$$

provided $(d_1, d_2) = 1$, and that $g(p^j) = 0$ for any prime p if $j \geq 2$.

Consider first the case $n = 3$. The coefficients $\tau(r_1, r_2)$ were computed by Proskurin [12] and by Bump and Hoffstein [4]. In particular, $\tau(m, 1) = 0$ unless m is a unit times a perfect cube, and $\tau(am^3, 1) = |m| \tau(a, 1)$ for $0 \neq a \in \mathcal{O}_S$. After some relabeling of variables, the series then becomes

$$D_j(s, \varphi) = \sum_{0 \neq d \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_l X_{lj}^{[d^{-1}]} g(d) b_l(m^3 d) |m|^{-s_1} |d|^{-s_1-1},$$

where $s_1 = 6s - 5/2$.

Suppose that φ is a Hecke eigenform at p with eigenvalue λ_p . Then if $(p, M) = 1$, the Fourier coefficients $b_l(m)$ satisfy a relation ([6], Cor. 3.3)

$$\begin{aligned} \lambda_p b_l(p^k M) = \\ b_l(p^{k-3} M) + b_l(p^{k+3} M) + |p|^{-1} \left(\frac{p}{M} \right)^{-k-1} g_{k+1}(p) \sum_i X_{il}^{[p^{-k-1}]} b_i(p^{1-k} M). \end{aligned}$$

Here the coefficients $b_l(m)$ are zero by definition if m is not in \mathcal{O}_S .

To analyze the contribution obtained from $D_j(s, \varphi)$ at p , we compute

$$(1 - \lambda_p p^{-s_1} + p^{-2s_1}) D_j(s, \varphi).$$

Since $g(d) = 0$ if $p^2 | d$, there are two nonzero contributions, from d such that $(d, p) = 1$ and from d such that $\text{ord}_p(d) = 1$. For the first contribution, the Hecke relation can be rewritten

$$\lambda_p b_l(p^{3k} M) = \begin{cases} b_l(p^{3(k-1)} M) + b_l(p^{3(k+1)} M) & \text{if } k \geq 1 \\ b_l(p^3 M) + |p|^{-1} \left(\frac{p}{M} \right)^{-1} g(p) \sum_i X_{il}^{[p^{-1}]} b_i(pM) & \text{if } k = 0 \end{cases}$$

where $(p, M) = 1$. Applying this it is not difficult to see that

$$\begin{aligned}
& (1 - \lambda_p p^{-s_1} + p^{-2s_1}) \times \\
& \sum_{\substack{0 \neq d \in \mathcal{O}_S / \mathcal{O}_S^\times \\ (d, p) = 1}} \sum_{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_l X_{lj}^{[d^{-1}]} g(d) b_l(m^3 d) |m|^{-s_1} |d|^{-s_1-1} = \\
& \sum_{\substack{0 \neq d \in \mathcal{O}_S / \mathcal{O}_S^\times \\ (d, p) = 1}} \sum_{\substack{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times \\ (m, p) = 1}} \sum_l X_{lj}^{[d^{-1}]} g(d) |m|^{-s_1} |d|^{-s_1-1} \\
& \left(b_l(m^3 d) - \left(\frac{p}{d}\right)^{-1} g(p) \sum_i X_{il}^{[p^{-1}]} b_i(pm^3 d) |p|^{-s_1-1} \right). \quad (9)
\end{aligned}$$

To analyze the terms with $\text{ord}_p(d) = p$, we make use of the Hecke relation

$$\lambda_p b_l(p^{3k+1} M) = \begin{cases} \bar{\chi}(p) b_l(p^{3(k-1)+1} M) + b_l(p^{3(k+1)+1} M) & \text{if } k \geq 1 \\ b_l(p^4 M) + |p|^{-1} \left(\frac{p}{M}\right) \bar{g}(p) \sum_i X_{il}^{[p^{-2}]} b_i(M) & \text{if } k = 0 \end{cases}$$

where $(p, M) = 1$. We also recall the properties ([6], Eqs. (3.11) and (3.12))

$$X_{jk}^{[ab]} = (b, a)_S \sum_\ell X_{j\ell}^{[a]} X_{\ell k}^{[b]} \quad a, b \in F^\times \quad (10)$$

and

$$X_{jk}^{[p^3]} = \delta_{j,k} \quad \text{Kronecker delta.} \quad (11)$$

Using the Hecke relations, Eqns. (10) (with $a = p^{-1}$, $b = d^{-1}$) and (11) and reordering, one finds that

$$\begin{aligned}
& (1 - \lambda_p p^{-s_1} + p^{-2s_1}) \times \\
& \sum_{\substack{0 \neq d \in \mathcal{O}_S / \mathcal{O}_S^\times \\ (d, p) = 1}} \sum_{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times} \sum_l X_{lj}^{[d^{-1}p^{-1}]} g(dp) b_j(m^3 dp) |m|^{-s_1} |dp|^{-s_1-1} = \\
& \sum_{\substack{0 \neq d \in \mathcal{O}_S / \mathcal{O}_S^\times \\ (d, p) = 1}} \sum_{\substack{0 \neq m \in \mathcal{O}_S / \mathcal{O}_S^\times \\ (m, p) = 1}} \sum_\ell X_{\ell j}^{[d^{-1}]} g(d) g(p) \left(\frac{p}{d}\right) \left(\frac{d}{p}\right) |m|^{-s_1} |d|^{-s_1-1} \\
& \left((d, p)_S \sum_l X_{\ell l}^{[p^{-1}]} b_l(pm^3 d) |p|^{-s_1-1} - |p|^{-2s_1-2} \left(\frac{p}{d}\right) \bar{g}(p) b_\ell(md^3) \right). \quad (12)
\end{aligned}$$

We may now combine the terms (9) and (12). Indeed, after using cubic reciprocity, we see that the last term on the right-hand side of (9) cancels a term on the right-hand-side of (12). Then iterating over the primes p of \mathcal{O}_S , we obtain the following Theorem.

Theorem 4. *Suppose that $n = 3$. Let*

$$D_j^*(s, \varphi) = \zeta(12s - 4) D_j(s, \varphi).$$

Then for each j ,

$$D_j^*(s, \varphi) = b_j(1) \prod_p (1 - \lambda_p |p|^{-s_1} + |p|^{-2s_1})^{-1}$$

with $s_1 = 6s - 5/2$.

Here for arbitrary $n \geq 2$, $\zeta(4ns - 2n + 2)$ is the normalizing factor of the Eisenstein series $E_\Theta(g, s, f_s)$.

When $n = 3$, the series considered by Bump and Hoffstein [6] is essentially of the form

$$\sum_l \sum_{m_1, m_2} \bar{\tau}(m_1, m_2) b_l(m_1) |m_1 m_2^2|^{-s}.$$

Though it represents the same Euler product (at $3s - 1$), the convolution (8) given here is visibly different.

To conclude this section, we briefly discuss the case $n = 4$. In this case, φ is an automorphic form on the *double* cover of $GL_2(F_S)$. In (8), the only coefficients $\tau(m, 1)$ of the theta function on the 4-fold cover of $GL_3(F_S)$ that are not zero are those that occur when each prime dividing m occurs to a power congruent to 0 or 1 modulo 4. Moreover, at prime powers we have

$$\tau(p^{4k}, 1) = |p|^k \quad \tau(p^{4k+1}, 1) = |p|^{k-1/2} \bar{g}(p).$$

Thus the series involves both the Fourier coefficients of φ at primes p , whose squares are related to the central values of the Shimura lift of φ twisted by a quadratic character modulo p , and quartic Gauss sums $g(p)$.

References

- [1] B. Brubaker, D. Bump and S. Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal graphs. *Ann. of Math.*, 173:1081–1120, 2011.

- [2] B. Brubaker and S. Friedberg. The Whittaker coefficients of metaplectic Eisenstein series. In preparation.
- [3] D. Bump and S. Friedberg. Metaplectic generating functions and Shimura integrals. *Proc. Symp. Pure Math.*, 66(2):1–17, 1999.
- [4] D. Bump and J. Hoffstein. Cubic metaplectic forms on $GL(3)$. *Invent. Math.* 84(3):481–505, 1986.
- [5] D. Bump and J. Hoffstein. Some Euler products associated with cubic metaplectic forms on $GL(3)$. *Duke Math. J.*, 53(4):1047–1072, 1986.
- [6] D. Bump and J. Hoffstein. On Shimura’s correspondence. *Duke Math. J.*, 55(3):661–691, 1987.
- [7] D. Ginzburg. Eulerian integrals for GL_n . *Proc. Symp. Pure Math.*, 75:203–223, 2006.
- [8] D. Ginzburg, S. Rallis and D. Soudry. The descent map from automorphic representations of $GL(n)$ to classical groups. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [9] J. Hoffstein. Eisenstein series and theta functions on the metaplectic group. *CRM Proc. Lecture Notes*, 1:65–104, 1993.
- [10] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.
- [11] I. Piatetski-Shapiro and S. Rallis. A new way to get Euler products. *J. Reine Angew. Math.*, 392:110–124, 1988.
- [12] N. Proskurin. Automorphic functions and the Bass-Milnor-Serre homomorphism, I, II. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 129:85–126, 127–163, 1983.
- [13] G. Shimura. On the holomorphy of certain Dirichlet series. *Proc. London Math. Soc.*, 31(1):79–98, 1975.
- [14] T. Suzuki. Distinguished representations of metaplectic groups. *Amer. J. Math.*, 120(4):723–755, 1988.

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL, MA 02467-3806, USA

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, RAMAT AVIV, TEL AVIV 6997801, ISRAEL